Analytical Solutions of 2D Incompressible Navier-Stokes Equations for Time Dependent Pressure Gradient

M.A.K.Azad, L.S. Andallah

Abstract- In this paper, we present analytical solutions of two dimensional incompressible Navier-Stokes equations (2D NSEs) for a time dependent exponentially decreasing pressure gradient term using Orlowski and Sobczyk transformation (OST) and Cole-Hopf transformation (CHT) . To find the solution of 2D diffusion equation we apply separation of variables method.

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1 INTRODUCTION

The Navier–Stokes equations are important governing equations in the fluid dynamics which describe the motion of fluid. These equations arise from applying Newton's second law to fluid motion, together with the assumption that the fluid feels forces due to pressure, viscosity and perhaps an external force. They are useful because they describe the physics of many things of academic and economic interest.

However, NSEs are nonlinear in nature and it is difficult to solve these equations analytically. In order to perform this task, some simplifications are elucidated, such as linearization or assumptions of weak nonlinearity, small fluctuations, discretization, etc. Despite the concentrated research on Navier Stokes equations, their universal solutions are not achieved. The full solutions of the three-dimensional NSEs remain one of the open problems in mathematical physics.

Computational Fluid Dynamics (CFD) approaches discritize the equations and   solve them   numerically. Although such numerical methods are successful, they are still expensive and there must be approximation errors associated with them. The development of high speed

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[1] Muhammad Abul Kalam Azad,Ph. D student, [2] Professor Dr. Laek Sazzad Andallah
akazad2034@gmail.com   andallahls@gmail.com

Department of Mathematics, Jahangirnagar University,Savar,Dhaka, Bangladesh.

computers eventually makes discretization methods more advanced and it enables the numerical treatment of turbulent flow.

The exact solutions for the NSEs can be obtained are of particular cases. Exact solutions on the other hand are very important for many reasons. They provide reference solutions to verify the approximate methods. In order to understand the non-linear phenomenon of NSEs, one needs to study 2D NSEs. Since these equations include the diffusion, advection and pressure gradient terms of the full 3D NSEs, these incorporate all the main mathematical features of the NSEs.

Applying OST we have reduced 2D NSEs to 2D viscous Burgers equations and we have solved Burgers equations analytically by using CHT. So a number of analytical and numerical studies on 2D NSEs and 2D viscous Burgers equations have been conducted to solve the governing equations analytically [1],[2],[3],[4],[5],[6],[7],[8],[9].

 Prof. Sanjay Mittal [1] studied on analytical solutions of simplified form of NSEs . Mohammad Mehedi Rashidi [2] presented new analytical solutions of the 3D NSEs. Maria Carmela Lombardo [3] studied on analytical solutions of the time dependent incompressible NSEs on an half plane . Hikmat G. Hasanov [4] found a method of 2D nonlinear Laplace transformation for solving the NSEs. Hui, W. H. [5] studied on the 2D incompressible viscous flow in which the local vorticity is proportional to the stream function perturbed by a uniform stream. Muhammad R. Mohyuddin et. el.[6] presented analytical solutions of 2D NSEs governing the unsteady incompressible flow. The solutions have been obtained using hondograph-Legendre transformation method. Vineet Kumar Srivastava, Ashutosh, Mohammad Tamsir [7]] derived a general analytical solutions of 3D homogeneous coupled unsteady non-linear generalized viscous Burgers' equations via CHT. C.A. J. Fletcher [8] have generated exact solutions of the 2D Burgers equations. T.X. Yan, L.S. Yue [9]   presented

variable separable solutions for the (2+1) –dimensional Burgers' equation.

In this paper, we reduce 2D NSEs into 2D coupled Burgers equations by applying OST. Then after applying CHT 2D Burgers equations will be reduced to 2D diffusion equation. By using separation of variables method we will solve diffusion equation. Then applying CHT and inverse OST we get the analytical solutions of 2D NSEs. One example has been carried out and their graphical representation is studied.

2 MATHEMATICAL FORMULATION

The dimensionalised governing equations of the fluid flow are given respectively by the

continuity equation

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0$$

x-momentum equation

$$\frac{\partial u^*}{\partial t^*} + u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} = -\frac{1}{\rho}\frac{\partial p^*}{\partial x^*} + \nu\left(\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{\partial^2 u^*}{\partial y^{*2}}\right)$$

y-momentum equation

$$\frac{\partial v^*}{\partial t^*} + u^* \frac{\partial v^*}{\partial x^*} + v^* \frac{\partial v^*}{\partial y^*} = -\frac{1}{\rho}\frac{\partial p^*}{\partial y^*} + \nu\left(\frac{\partial^2 v^*}{\partial x^{*2}} + \frac{\partial^2 v^*}{\partial y^{*2}}\right)$$

Where u and v are the velocity components in the x and y directions respectively, p is the pressure, ρ is the constant density and υ is the kinematic viscosity.

Using the dimensionless definitions in [10],

$$t = \frac{t^* U}{h}, x = \frac{x^*}{h}, y = \frac{y^*}{h}, u = \frac{u^*}{U}, v = \frac{v^*}{U}, p = \frac{p^*}{\rho U^2}$$

The dimensionalised governing equations are then converted into the non-dimensional form 2D NSEs as

$$u_t + uu_x + vu_y = -p_x + \frac{1}{\text{Re}}\left(u_{xx} + u_{yy}\right) \qquad (1)$$

$$v_t + uv_x + vv_y = -p_y + \frac{1}{\text{Re}}\left(v_{xx} + v_{yy}\right) \qquad (2)$$

$$u_x + v_y = 0 \qquad (3)$$

where $\dfrac{1}{\text{Re}} = \dfrac{\nu}{Uh}, -p_x = f(t) = e^{-at-b}; -p_y = g(t) = e^{-ct-d}$. Re is the Reynolds number.

As a result " (1)" can be rewritten as

$$u_t + uu_x + vu_y = f(t) + \frac{1}{\text{Re}}\left(u_{xx} + u_{yy}\right) \qquad (4)$$

" (2)" can be written as

$$v_t + uv_x + vv_y = g(t) + \frac{1}{\text{Re}}\left(u_{xx} + u_{yy}\right) \qquad (5)$$

Then we apply OST [11] as

$$x' = x - \phi(t), y' = y - \psi(t), t' = t, u'(x',y',t') = u(x,y,t) - W(t), v' = v(x,y,t) - W'(t). \qquad (6)$$

With $W(t) = \int_0^t f(\tau)d\tau, \phi(t) = \int_0^t W(\tau)d\tau \qquad (7)$

And $W'(t) = \int_0^t g(\tau)d\tau, \psi(t) = \int_0^t W'(\tau)d\tau \qquad (8)$

Here $W(t) = \int_0^t f(\tau)d\tau$

$$= \int_0^t e^{-a\tau-b}d\tau$$

$$= \left[\frac{e^{-a\tau-b}}{-a}\right]_0^t$$

$$= \frac{1}{a}\left(e^{-b} - e^{-at-b}\right).$$

Again $\phi(t) = \int_0^t W(\tau)d\tau$

$$= \int_0^t \frac{1}{a}\left(e^{-b} - e^{-a\tau-b}\right)d\tau$$

$$= \frac{1}{a}\int_0^t \left(e^{-b} - e^{-a\tau-b}\right)d\tau$$

$$= \frac{1}{a}\left[\tau e^{-b} + \frac{e^{-a\tau-b}}{a}\right]_0^t$$

$$= \frac{1}{a}\left[te^{-b} + \frac{e^{-at-b}}{a} - \frac{e^{-b}}{a}\right]$$

Now $W'(t) = \int_0^t g(\tau)d\tau$

$$= \int_0^t e^{-c\tau-d}d\tau$$

$$= \left[\frac{e^{-c\tau-d}}{-c}\right]_0^t$$

$$= \frac{1}{c}\left(e^{-d} - e^{-ct-d}\right).$$

$$= \frac{\partial u'}{\partial x'}.$$

Again $\psi(t) = \int_0^t W'(\tau)d\tau$

$$= \int_0^t \frac{1}{c}\left(e^{-d} - e^{-c\tau-d}\right)d\tau$$

$$= \frac{1}{c}\int_0^t \left(e^{-d} - e^{-c\tau-d}\right)d\tau$$

$$= \frac{1}{c}\left[\tau e^{-d} + \frac{e^{-c\tau-d}}{c}\right]_0^t$$

$$= \frac{1}{c}\left[te^{-d} + \frac{e^{-c\tau-d}}{c} - \frac{e^{-d}}{c}\right]$$

Again $\dfrac{\partial u}{\partial y} = \dfrac{\partial}{\partial y}\left(u' + \dfrac{1}{a}e^{-b} - \dfrac{1}{a}e^{-at-b}\right)$

$$= \frac{\partial u'}{\partial y}$$

$$= \frac{\partial u'}{\partial x'}\cdot\frac{\partial x'}{\partial y} + \frac{\partial u'}{\partial y'}\cdot\frac{\partial y'}{\partial y} + \frac{\partial u'}{\partial t'}\cdot\frac{\partial t'}{\partial y}$$

$$= \frac{\partial u'}{\partial y'}.$$

Thus we get $x' = x - \dfrac{1}{a}te^{-b} - \dfrac{1}{a^2}e^{-at-b} + \dfrac{1}{a^2}e^{-b}$ (9)

$$y' = y - \frac{1}{c}te^{-d} - \frac{1}{c^2}e^{-ct-d} + \frac{1}{c^2}e^{-d}$$ (10)

$$t' = t$$ (11)

$$u'(x', y', t') = u(x, y, t) - \frac{1}{a}e^{-b} + \frac{1}{a}e^{-at-b}.$$ (12)

$$v'(x', y', t') = v(x, y, t) - \frac{1}{c}e^{-d} + \frac{1}{c}e^{-ct-d}.$$ (13)

Also $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial x}\right)$

$$= \frac{\partial}{\partial x}\left(\frac{\partial u'}{\partial x'}\right)$$

$$= \frac{\partial}{\partial x'}\left(\frac{\partial u'}{\partial x'}\right)\cdot\left(\frac{\partial x'}{\partial x}\right) + \frac{\partial}{\partial y'}\left(\frac{\partial u'}{\partial x'}\right)\cdot\left(\frac{\partial y'}{\partial x}\right) + \frac{\partial}{\partial t'}\left(\frac{\partial u'}{\partial x'}\right)\cdot\left(\frac{\partial t'}{\partial x}\right)$$

$$u'(x', y', t') = u(x, y, t) - \frac{1}{a}e^{-b}.$$

$$= \frac{\partial^2 u'}{\partial x'^2}.$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)$$

$$= \frac{\partial}{\partial y}\left(\frac{\partial u'}{\partial y'}\right)$$ (13)

$$= \frac{\partial}{\partial x'}\left(\frac{\partial u'}{\partial y'}\right)\cdot\left(\frac{\partial x'}{\partial y}\right) + \frac{\partial}{\partial y'}\left(\frac{\partial u'}{\partial y'}\right)\cdot\left(\frac{\partial y'}{\partial y}\right) + \frac{\partial}{\partial t'}\left(\frac{\partial u'}{\partial y'}\right)\cdot\left(\frac{\partial t'}{\partial y}\right)$$

$$= \frac{\partial^2 u'}{\partial y'^2}.$$

Now $\dfrac{\partial u}{\partial t} = \dfrac{\partial}{\partial t}\left(u' + \dfrac{1}{a}e^{-b} - \dfrac{1}{a}e^{-at-b}\right)$

$$= \frac{\partial u'}{\partial t} + e^{-at-b}$$

$$= \frac{\partial u'}{\partial x'}\cdot\frac{\partial x'}{\partial t} + \frac{\partial u'}{\partial y'}\cdot\frac{\partial y'}{\partial t} + \frac{\partial u'}{\partial t'}\cdot\frac{\partial t'}{\partial t} + e^{-at-b}$$

$$= \frac{\partial u'}{\partial x'}\cdot\left(-\frac{1}{a}e^{-b} + \frac{1}{a}e^{-at-b}\right) + \frac{\partial u'}{\partial y'}\cdot\left(-\frac{1}{c}e^{-d} + \frac{1}{c}e^{-ct-d}\right) + \frac{\partial u'}{\partial t'} + e^{-at-b}.$$

$$\frac{\partial v}{\partial t} = \frac{\partial}{\partial t}\left(v' + \frac{1}{c}e^{-d} - \frac{1}{c}e^{-ct-d}\right)$$

$$= \frac{\partial v'}{\partial t} + e^{-ct-d}$$

$$= \frac{\partial v'}{\partial x'}\cdot\frac{\partial x'}{\partial t} + \frac{\partial v'}{\partial y'}\cdot\frac{\partial y'}{\partial t} + \frac{\partial v'}{\partial t'}\cdot\frac{\partial t'}{\partial t} + e^{-ct-d}$$

$$= \frac{\partial v'}{\partial x'}\cdot\left(-\frac{1}{a}e^{-b} + \frac{1}{a}e^{-at-b}\right) + \frac{\partial v'}{\partial y'}\cdot\left(-\frac{1}{c}e^{-d} + \frac{1}{c}e^{-ct-d}\right) + \frac{\partial v'}{\partial t'} + e^{-ct-d}.$$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(u' + \frac{1}{a}e^{-b} - \frac{1}{a}e^{-at-b}\right)$$

$$= \frac{\partial u'}{\partial x}$$

$$= \frac{\partial u'}{\partial x'}\cdot\frac{\partial x'}{\partial x} + \frac{\partial u'}{\partial y'}\cdot\frac{\partial y'}{\partial x} + \frac{\partial u'}{\partial t'}\cdot\frac{\partial t'}{\partial x}$$

$$\frac{\partial v}{\partial x} = \frac{\partial}{\partial x}\left(v' + \frac{1}{c}e^{-d} - \frac{1}{c}e^{-ct-d}\right)$$

$$= \frac{\partial v'}{\partial x}$$

$$= \frac{\partial v'}{\partial x'} \cdot \frac{\partial x'}{\partial x} + \frac{\partial v'}{\partial y'} \cdot \frac{\partial y'}{\partial x} + \frac{\partial v'}{\partial t'} \cdot \frac{\partial t'}{\partial x}$$

$$= \frac{\partial v'}{\partial x'}.$$

$$\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}\left(v' + \frac{1}{c}e^{-d} - \frac{1}{c}e^{-ct-d}\right)$$

$$= \frac{\partial v'}{\partial y}$$

$$= \frac{\partial v'}{\partial x'} \cdot \frac{\partial x'}{\partial y} + \frac{\partial v'}{\partial y'} \cdot \frac{\partial y'}{\partial y} + \frac{\partial v'}{\partial t'} \cdot \frac{\partial t'}{\partial y}$$

$$= \frac{\partial v'}{\partial y'}.$$

Also $\frac{\partial^2 v}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right)$

$$= \frac{\partial}{\partial x}\left(\frac{\partial v'}{\partial x'}\right)$$

$$= \frac{\partial}{\partial x'}\left(\frac{\partial v'}{\partial x'}\right) \cdot \left(\frac{\partial x'}{\partial x}\right) + \frac{\partial}{\partial y'}\left(\frac{\partial v'}{\partial x'}\right) \cdot \left(\frac{\partial y'}{\partial x}\right) + \frac{\partial}{\partial t'}\left(\frac{\partial v'}{\partial x'}\right) \cdot \left(\frac{\partial t'}{\partial x}\right)$$

$$= \frac{\partial^2 v'}{\partial x'^2} \cdot 1 + \frac{\partial^2 v'}{\partial y'\partial x'} \cdot 0 + \frac{\partial}{\partial t'}\left(\frac{\partial u'}{\partial x'}\right) \cdot 0$$

$$= \frac{\partial^2 v'}{\partial x'^2}.$$

$$\frac{\partial^2 v}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial y}\right)$$

$$= \frac{\partial}{\partial y}\left(\frac{\partial v'}{\partial y'}\right)$$

$$= \frac{\partial}{\partial x'}\left(\frac{\partial v'}{\partial y'}\right) \cdot \left(\frac{\partial x'}{\partial y}\right) + \frac{\partial}{\partial y'}\left(\frac{\partial v'}{\partial y'}\right)\left(\frac{\partial y'}{\partial y}\right) + \frac{\partial}{\partial t'}\left(\frac{\partial v'}{\partial y'}\right) \cdot \left(\frac{\partial t'}{\partial y}\right)$$

$$= \frac{\partial^2 v'}{\partial x'\partial y'^2} \cdot 0 + \frac{\partial^2 v'}{\partial y'^2} \cdot 1 + \frac{\partial^2 v'}{\partial t'\partial y'} \cdot 0$$

$$= \frac{\partial^2 v'}{\partial y'^2}.$$

Substituting the transformed derivatives in "(2)" , we get

$$u'_{x'}\left(-\frac{1}{a}e^{-b} + \frac{1}{a}e^{-at-b}\right) + u'_{y'}\left(-\frac{1}{c}e^{-d} + \frac{1}{c}e^{-ct-d}\right)$$

$$+ u'_{t'} + e^{-at-b} + \left(u' + \frac{1}{a}e^{-b} - \frac{1}{a}e^{-at-b}\right)u'_{x'}$$

$$+ \left(v' + \frac{1}{c}e^{-d} - \frac{1}{c}e^{-ct-d}\right)u'_{y'} = \frac{1}{\text{Re}}\left(u'_{x'x'} + u'_{y'y'}\right) + e^{-at-b}$$

$$\Rightarrow u'_{t'} + u'u'_{x'} + v'u'_{y'} = \frac{1}{\text{Re}}\left(u'_{x'x'} + u'_{y'y'}\right)$$

$$\Rightarrow u'_{t'} + u'u'_{x'} + v'u'_{y'} = \frac{1}{\text{Re}}\left(u'_{x'x'} + u'_{y'y'}\right) \qquad (14)$$

Similarly, substituting the transformed derivatives in "(3)" , we get

$$v'_{t'} + u'v'_{x'} + v'v'_{y'} = \frac{1}{\text{Re}}\left(v'_{x'x'} + v'_{y'y'}\right) \qquad (15)$$

From " (4)" , we obtain

$$u'_{x'} + v'_{y'} = 0 \qquad (16)$$

Again, we know that the non-dimensional form of 2D Burgers' equations are

$$\frac{\partial u^*}{\partial t^*} + u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*} = \frac{1}{\text{Re}}\left(\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{\partial^2 u^*}{\partial y^{*2}}\right). \qquad (17)$$

$$\frac{\partial v^*}{\partial t^*} + u^*\frac{\partial v^*}{\partial x^*} + v^*\frac{\partial v^*}{\partial y^*} = \frac{1}{\text{Re}}\left(\frac{\partial^2 v^*}{\partial x^{*2}} + \frac{\partial^2 v^*}{\partial y^{*2}}\right). \qquad (18)$$

So, " (14)", " (15)" are the transformed 2D Navier-Stokes equations after applying Orlowski and Sobczyk transformation and these are analogous to non-dimensional form of 2D Burgers equation " (17)", " (18)".

Now we need to solve "(17)" with initial conditions
$u'(x', y', 0) = u_0(x', y'), v'(x', y', 0) = v'_0(x', y')$
$where -\infty < x' < \infty, -\infty < y' < \infty.$

"(14)" and "(15)" can be linearized by the CHT [7],[8],[12],

$$u'(x', y', t') = -\frac{2}{\text{Re}}\frac{\phi'_{x'}}{\phi'} \qquad (19)$$

And
$$v'(x', y', t') = -\frac{2}{\text{Re}}\frac{\phi'_{y'}}{\phi'} \qquad (20)$$

We perform the transformation in some steps.

$$u' = \phi'_{x'}, \; v' = \phi'_{y'}.$$

First let us assume

With this transformation (17) becomes

$$\phi'_{x't'} + \phi'_{x'}\phi'_{x'x'} + \phi'_{y'}\phi'_{x'y'} = \frac{1}{\text{Re}}\left(\phi'_{x'x'x'} + \phi'_{x'y'y'}\right)$$

$$\Rightarrow \phi'_{x't'} + \left(\frac{1}{2}\phi'^2_{x'}\right)_{x'} + \left(\frac{1}{2}\phi'^2_{y'}\right)_{x'} = \frac{1}{\text{Re}}\left(\phi'_{x'x'} + \phi'_{y'y'}\right)_{x'}$$

$$\Rightarrow \left(\phi'_{t'} + \frac{1}{2}\phi'^2_{x'} + \frac{1}{2}\phi'^2_{y'}\right)_{x'} = \left\{\frac{1}{\text{Re}}\left(\phi'_{x'x'} + \phi'_{y'y'}\right)\right\}_{x'}.$$

Now integrating the transformed equation w.r.t. x', then we have

$$\phi'_{t'} + \frac{1}{2}\phi'^2_{x'} + \frac{1}{2}\phi'^2_{y'} = \frac{1}{\text{Re}}\left(\phi'_{x'x'} + \phi'_{y'y'}\right) \qquad (21)$$

Then we make the transformation

$$\phi = -\frac{2}{\text{Re}}\ln\psi$$

$$\phi_{t'} = -\frac{2}{\text{Re}}\frac{\psi_{t'}}{\psi}$$

$$\phi'_{x'} = -\frac{2}{\text{Re}}\frac{\psi'_{x'}}{\psi'}$$

$$\phi'_{y'} = -\frac{2}{\text{Re}}\frac{\psi'_{y'}}{\psi'}$$

$$\phi'_{x'x'} = -\frac{2}{\text{Re}}\frac{\psi'\psi'_{x'x'} - \psi'_{x'}\psi'_{x'}}{\psi^2} = -\frac{2}{\text{Re}}\frac{\psi'\psi'_{x'x'} - \psi'^2_{x'}}{\psi'^2}$$

$$\phi'_{y'y'} = -\frac{2}{\text{Re}}\frac{\psi'\psi'_{y'y'} - \psi'_{y'}\psi'_{y'}}{\psi^2} = -\frac{2}{\text{Re}}\frac{\psi'\psi'_{y'y'} - \psi'^2_{y'}}{\psi'^2}$$

Substituting these derivatives in (21), we obtain

$$-\frac{2}{\text{Re}}\frac{\psi'_{t'}}{\psi'} + \frac{1}{2}\cdot\frac{4}{\text{Re}^2}\frac{\psi'^2_{x'}}{\psi'^2} + \frac{1}{2}\cdot\frac{4}{\text{Re}^2}\frac{\psi'^2_{y'}}{\psi'^2}$$

$$= \frac{1}{\text{Re}}\left(-\frac{2}{\text{Re}}\frac{\psi'\psi'_{x'x'} - \psi'^2_{x'}}{\psi'^2} - \frac{2}{\text{Re}}\frac{\psi'\psi'_{y'y'} - \psi'^2_{y'}}{\psi'^2}\right)$$

$$\Rightarrow \psi'_{t'} = \frac{1}{\text{Re}}\left(\psi'_{x'x'} + \psi'_{y'y'}\right) \qquad (22)$$

which is the well-known second order PDE called heat or diffusion equation [10,11].

From equation (19) we get

$$u' = -\frac{2}{\text{Re}}\left(\ln\psi'\right)_{x'}$$

$$\Rightarrow \left(\ln\psi'\right)_{x'} = -\frac{\text{Re}}{2}u'$$

$$\Rightarrow \ln\psi' = -\int\frac{\text{Re}}{2}u'dx' + \ln C$$

$$\Rightarrow \ln\left(\frac{\psi'}{C}\right) = -\frac{\text{Re}}{2}\int u'\,dx'$$

$$\Rightarrow \frac{\psi'}{C} = e^{-\frac{\text{Re}}{2}\int u'dx'}$$

$$\Rightarrow \psi'(x', y', t') = Ce^{-\frac{\text{Re}}{2}\int u'dx'} \qquad (23)$$

It is clear from (19) that multiplying ψ' by a constant does not affect u', so we can write "(23)" as

$$\psi'(x', y', t') = e^{-\frac{\text{Re}}{2}\int_0^{x'} u'(z,t)dz} \qquad (24)$$

For $t' = 0$, equation (24) gives

$$\psi'(x', y', 0) = e^{-\frac{\text{Re}}{2}\int_0^{x'} u'(z,0)dz}$$

$$\Rightarrow \psi'(x', y', 0) = e^{-\frac{\text{Re}}{2}\int_0^{x'} u'_0(z)dz} = \psi_0(x', y')(let)$$

Thus we can say that 2D Burgers equation can be reduced to 2D diffusion equation by using CHT.

Again, for "(18)" we perform the transformation in some steps.

$$u' = \phi'_{x'}, \; v' = \phi'_{y'}.$$

Let us assume

With this transformation (18) becomes

$$\phi'_{y't'} + \phi'_{x'}\phi_{y'x'} + \phi'_{y'}\phi'_{y'y'} = \frac{1}{\text{Re}}\left(\phi'_{y'x'x'} + \phi'_{y'y'y'}\right)$$

$$\Rightarrow \phi'_{y't'} + \left(\frac{1}{2}\phi'^2_{x'}\right)_{y'} + \left(\frac{1}{2}\phi'^2_{y'}\right)_{y'} = \frac{1}{\text{Re}}\left(\phi'_{x'x'} + \phi'_{y'y'}\right)_{y'}$$

$$\Rightarrow \left(\phi'_{t'} + \frac{1}{2}\phi'^2_{x'} + \frac{1}{2}\phi'^2_{y'}\right)_{y'} = \left\{\frac{1}{\text{Re}}\left(\phi'_{x'x'} + \phi'_{y'y'}\right)\right\}_{y'}.$$

Now integrating the transformed equation w.r.t. y', then we have

$$\phi'_t + \frac{1}{2}\phi'^2_{x'} + \frac{1}{2}\phi'^2_{y'} = \frac{1}{Re}\left(\phi'_{x'x'} + \phi'_{y'y'}\right) \tag{25}$$

Then we make the transformation

$$\phi' = -\frac{2}{Re}\ln\psi'$$

$$\phi'_{t'} = -\frac{2}{Re}\frac{\psi'_{t'}}{\psi'}$$

$$\phi'_{x'} = \frac{2}{Re}\frac{\psi'_{x'}}{\psi'}$$

$$\phi'_{y'} = \frac{2}{Re}\frac{\psi'_{y'}}{\psi'}$$

$$\phi'_{x'x'} = -\frac{2}{Re}\frac{\psi'\psi'_{x'x'} - \psi'_{x'}\psi'_{x'}}{\psi'^2} = -\frac{2}{Re}\frac{\psi'\psi'_{x'x'} - \psi'^2_{x'}}{\psi'^2}$$

$$\phi'_{y'y'} = -\frac{2}{Re}\frac{\psi'\psi'_{y'y'} - \psi'_{y'}\psi'_{y'}}{\psi'^2} = -\frac{2}{Re}\frac{\psi'\psi'_{y'y'} - \psi'^2_{y'}}{\psi'^2}$$

Substituting these derivatives in "(25)", we obtain

$$\psi'_{t'} = \frac{1}{Re}\left(\psi'_{x'x'} + \psi'_{y'y'}\right) \tag{26}$$

which is the well-known second order PDE called heat or diffusion equation and it is exactly same as "(22)".

From equation (19) we get

$$v' = -\frac{2}{Re}\left(\ln\psi'\right)_{y'}$$

$$\Rightarrow \left(\ln\psi'\right)_{y'} = -\frac{Re}{2}v'$$

$$\Rightarrow \ln\psi' = -\int\frac{Re}{2}v'dy' + \ln C$$

$$\Rightarrow \ln\left(\frac{\psi'}{C}\right) = -\frac{Re}{2}\int v'\,dy'$$

$$\Rightarrow \psi'(x',y',t') = Ce^{-\frac{Re}{2}\int v'(x',y',t')dy'} \tag{27}$$

It is clear from (19) that multiplying ψ' by a constant does not affect v', so we can write "(27)" as

$$\psi(x',y',t') = e^{-\frac{Re}{2}\int_0^{y'} v'(z,t)dz} \tag{28}$$

For $t' = 0$, equation (28) gives

$$\psi'(x',y',0) = e^{-\frac{Re}{2}\int_0^{y'} v'(z,0)dz}$$

$$\Rightarrow \psi'(x',y',0) = e^{-\frac{Re}{2}\int_0^{y'} v'_0(z)dz} = \psi'_0(x',y')(let)$$

which is the well-known second order PDE called heat or diffusion equation and it is exactly same as "(22)".

Consider a general solution of "(22)" of the form similar analogy to [7],[13]

$$\psi'(x',y',t') = a_1 + a_2 x' + a_3 y' + a_4 x'y' + X(x')Y(y')T(t') \tag{29}$$

which is the sum of the solution $\psi'_1(x',y',t') = a_1 + a_2 x' + a_3 y' + a_4 x'y'$ and the separable solution $\psi'_2(x',y',t') = X(x')Y(y')T(t')$ where a_1, a_2, a_3, a_4 are arbitrary constants and X,Y,T are functions of x', y', t' respectively. Then "(22)" becomes

$$XYT' = c\left(X''YT + XY''T\right)$$

Dividing by cXYT on both sides, we get

$$\frac{T'}{cT} = \frac{X''}{X} + \frac{Y''}{Y}$$

Since the left side is a function of t' alone, while the right side is a function of x' and y', we see that each side must be a constant, say $-\lambda^2$ (which is needed for boundedness). Thus

$$T' + c\lambda^2 T' = 0 \tag{30}$$

$$\frac{X''}{X} + \frac{Y''}{Y} = -\lambda^2 \tag{31}$$

"(31)" can be written as

$$\frac{X''}{X} = -\frac{Y''}{Y} - \lambda^2$$

And since the left side depends only on x' while the right side depends only on y' each side must be a constant, say -μ^2. Thus we obtain

$$X'' + \mu^2 X = 0 \tag{32}$$

$$Y'' + \left(\lambda^2 - \mu^2\right)Y = 0 \tag{33}$$

Where $\lambda^2 - \mu^2 = \alpha^2$

Solutions to "(32)", "(33)" and "(30)" are given by

$$X = b_1 \cos \mu x' + c_1 \sin \mu x',$$
$$Y = b_2 \cos \alpha y' + c_2 \sin \alpha y'$$
$$T = b_3 e^{-c\lambda^2 t'}$$

It follows that a solution to "(22)" is given by

$$\psi_2'(x', y', t') = (b_1 \cos \mu x' + c_1 \sin \mu x')(b_2 \cos \alpha y' + c_2 \sin \alpha y')(b_3 e^{-c\lambda^2 t'}) \quad (35)$$

Thus the general solution of 2D diffusion equation is given by

$$\psi'(x', y', t') = a_1 + a_2 x' + a_3 y' + a_4 x' y' \\ + (b_1 \cos \mu x' + c_1 \sin \mu x')(b_2 \cos \alpha y' + c_2 \sin \alpha y')(b_3 e^{-c\lambda^2 t'}) \quad (36)$$

By using CHT we obtain the general solution of 2D viscous Burgers equations as

$$u'(x', y', t') = -\frac{2}{Re}\left(\frac{a_2 + a_4 y' + (-b_1 \mu \sin \mu x' + c_1 \mu \cos \mu x')(b_2 \cos \alpha y' + c_2 \sin \alpha y')(b_3 e^{-c\lambda^2 t'})}{a_1 + a_2 x' + a_3 y' + a_4 x' y' + (b_1 \cos \mu x' + c_1 \sin \mu x')(b_2 \cos \alpha y' + c_2 \sin \alpha y')(b_3 e^{-c\lambda^2 t'})}\right)$$
$$(37)$$

$$v'(x', y', t') = -\frac{2}{Re}\left(\frac{a_3 + a_4 x' + (b_1 \cos \mu x' + c_1 \sin \mu x')(-b_2 \sin \alpha y' + c_2 \cos \alpha y')(b_3 \alpha e^{-c\lambda^2 t'})}{a_1 + a_2 x + a_3 y + a_4 xy + (b_1 \cos \mu x' + c_1 \sin \mu x')(b_2 \cos \alpha y' + c_2 \sin \alpha y')(b_3 e^{-c\lambda^2 t'})}\right)$$
$$(38)$$

Now applying inverse OST we get the analytical solutions of 2D NSEs as

$$u(x, y, t) = \frac{1}{a}e^{-b} - \frac{1}{a}e^{-at-b} + u'(x', y', t') \quad (39)$$

$$v(x, y, t) = \frac{1}{c}e^{-d} - \frac{1}{a}e^{-ct-d} + v'(x', y', t') \quad (40)$$

where x′,y′,t′,u′,v′ will be determined from equations (9)-(13).

4 DISCUSSION

4.1 Problem 1

In this problem the computational domain is $\Omega = \left\{(x', y') : 0 \le x' \le 1, 0 \le y' \le 1\right\}$.

The exact solutions of Burgers equations (14) and (15) can be generated by using the CHT [7], [8] ,[13] which are:

$$u'(x', y', t') = -\frac{4\pi e^{-\frac{5\pi^2 t'}{Re}} \cos(2\pi x')\sin(\pi y')}{Re\left(2 + e^{-\frac{5\pi^2 t'}{Re}} \sin(2\pi x')\sin(\pi y')\right)}$$
$$(34)$$

$$v'(x', y', t') = -\frac{2\pi e^{-\frac{5\pi^2 t'}{Re}} \sin(2\pi x')\cos(\pi y')}{Re\left(2 + e^{-\frac{5\pi^2 t'}{Re}} \sin(2\pi x')\sin(\pi y')\right)}$$

Initial conditions:

$$u'(x', y', 0) = -\frac{4\pi \cos(2\pi x')\sin(\pi y')}{Re(2 + \sin(2\pi x')\sin(\pi y'))}, (x', y') \in \Omega$$

$$v'(x', y', t') = -\frac{2\pi \sin(2\pi x')\cos(\pi y')}{Re(2 + \sin(2\pi x')\sin(\pi y'))}, (x', y') \in \Omega$$

Boundary conditions:

$$u'(0, y', t') = -\frac{2\pi e^{-\frac{5\pi^2 t'}{Re}} \sin(\pi y')}{Re}, \quad t' \ge 0;$$

$$u'(1, y', t') = -\frac{2\pi e^{-\frac{5\pi^2 t'}{Re}} \sin(\pi y')}{Re}, \quad t' \ge 0;$$

$$u'(x', 0, t') = 0, \quad t' \ge 0; \quad u'(x', 1, t') = 0, \quad t' \ge 0;$$

$$v'(0, y', t') = 0, \quad t' \ge 0; \quad v'(1, y', t') = 0, \quad t' \ge 0;$$

$$v'(x', 0, t') = -\frac{\pi e^{-\frac{5\pi^2 t'}{Re}} \sin(2\pi x')}{Re}, \quad t' \ge 0;$$

$$v'(x', 1, t') = \frac{\pi e^{-\frac{5\pi^2 t'}{Re}} \sin(2\pi x')}{Re}, \quad t' \ge 0;$$

The solutions for u′ and v′ are evaluated from 2D Burgers′ equations and then by using IOST solutions of NSEs are computed and plotted in fig. 1 and fig. 2.
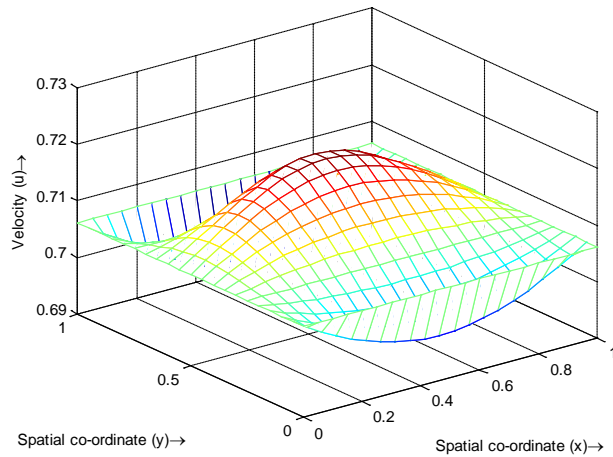
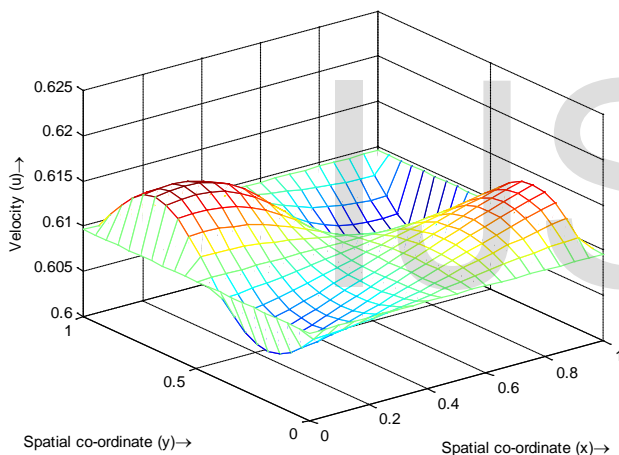Fig.1. The exact value of u at 21×21 grids for Re= 500 and at time level t = 1.0 with 501 time steps.

Fig.2. The exact value of v at 21×21 grids for Re= 500 and at time level t = 1.0 with 501 time steps.

6 CONCLUSION

In this study, we have shown how to solve analytical solutions of 2D NSEs by using OST and CHT with the help of separation of variables method. By using same method one can easily find out the analytical solutions 3D and 1D NSE . The method is simple and can be used to find general solution of the governing equations when we set initial and boundary conditions.

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